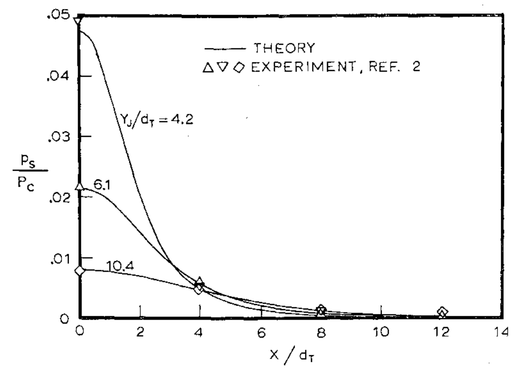

Fig. 2 Pressure distributions on the lunar surface for flow from the sonic nozzle

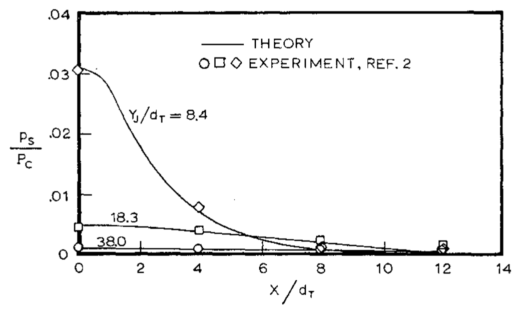

Fig. 3 Pressure distributions on the lunar surface for flow from the conical nozzle

For an ideal gas

$$\frac{P_{T2}}{p_1} = \left[\frac{(\gamma + 1)M_1^2}{2}\right]^{[\gamma/(\gamma-1)]} \left[\frac{\gamma + 1}{2\gamma M_1^2 + 1 - \gamma}\right]^{[1/(\gamma-1)]}$$

5) The other surface properties are calculated by noting that the surface entropy is constant and equal to the lunar-surface stagnation point value.

Reference 2 presents results from an experimental study to determine the effects of an exhaust plume on a simulated lunar surface. Lunar-surface pressure distributions for flow from cold air jets were measured at various descent heights. Several nozzles were used; however, only results for a single sonic nozzle and a single 4:1, 15° conical nozzle ($M_j = 2.94$) are used here. Figures 2 and 3 show experimental surface pressure distributions obtained using these two nozzles.

The theoretical values shown in these figures were calculated using the method presented here. A recently completed method of characteristics program[3] was used to calculate the jet exhaust plume. In order to obtain answers as accurately as possible, the shock shapes were obtained from schlieren photographs taken during the experiments of Ref. 2.

Referring to Figs. 2 and 3, it can be seen that there is excellent agreement between theory and experiment. If the shock shape had been assumed to lie on the lunar surface, the theoretical pressures would have been lower. This assumption was made in Ref. 2 and explains why the theoretical lunar pressures at the centerline of the jet were lower than the experimental values.

It is of interest to compare results with those calculated using the method of Ref. 1. Table 1 shows a comparison of centerline pressures for the conical nozzle. As can be seen, the method of Ref. 1 accurately predicts the centerline pressure at the larger values of descent height. At the lowest value the pressure is underpredicted; however, that method is expected to become less accurate as the descent height decreases.

Table 1 Comparison of centerline pressures for flow from the conical nozzle

	$(p_s/P_c)_{x/d_T=0}$		
Y_J/d_T	Theory this note	Theory Ref. 1	Experiment Ref. 2
8.4	0.0305	0.0189	0.0300
18.3	0.0047	0.0040	0.0045
38.0	0.0009	0.0009	0.0009

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Techniques for the Derivation of Element Stiffness Matrices

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Reference 1 has given a matrix formulation of the "unit load theorem" approach to the derivation of structural element stiffness properties. The present note shows that there are two alternative approaches to such determinations and presents a matrix formulation of both. One approach is based on a direct formulation of the desired relationships without recourse to work or strain energy principles, whereas the other is an application of the principle of virtual displacements. A brief comparison of all three alternatives also will be given. A detailed derivation and critical review of the three approaches appears in Ref. 2.

The notation of Ref. 1 will be retained to the extent possible. Consider first the "direct formulation" technique, the general concepts of which apply to derivations based on assumed stress or strain distributions or assumed displacements. To retain correspondence with Ref. 1, which restricts its attention to assumed stress distributions, one begins by considering a stress vector $\{\mathbf{\sigma}\}$ ($= \{\mathbf{\sigma}_{xx} \ldots \mathbf{\tau}_{zx}\}$), the terms of which are approximated by functions whose coefficients are the constants $\{k_1\}$. These assumed functional relationships for the stresses can be written in matrix form as

$$\{\mathbf{\sigma}\} = [\bar{U}]\{k_1\} \qquad (1)$$

where the terms of $[\bar{U}]$ are the variables in these relationships and are generally dimensional variables.

By use of the appropriate stress-strain relationships and subsequent integration of the strain-displacement equations, relationships for the displacements Δ are obtained in terms of the coefficients (k_1,k_2), where the added coefficients k_2 pertain to rigid body motion terms in the displacement functions. Evaluation of these displacement relationships at the points where stress resultants will be assumed to act provides the following set of algebraic equations (with $\{k\} = \{k_1\!:\!k_2\}$):

$$\{\Delta\} = [B]\{k\} \qquad (2)$$

hence

$$\{k\} = [B]^{-1}\{\Delta\} \qquad (3)$$

Since $[B]$ must be a square matrix, it is apparent that the present development is limited to cases where the coefficients (k_1) of the assumed stress distribution, plus the coefficients

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(k_2) associated with rigid body motion, are equal in number to the independent element displacements.

For conformability in subsequent matrix operations, the matrix $[\bar{U}]$ is expanded with columns of zeros to accommodate the coefficients $\{k_2\}$ as well as $\{k_1\}$ and is redesignated as $[U]$. Thus, Eq. (1) can be restated as

$$\{\mathbf{\delta}\} = [U]\{k\} \tag{1a}$$

Through integration of the stresses over the surface of the element, stress resultants $\{p\}$ at the chosen points can be defined in terms of the coefficients $\{k\}$:

$$\{p\} = [V]\{k\} \tag{4}$$

The matrix $[V]$ also contains sufficient zero columns to accommodate the coefficients $\{k_2\}$ and therefore the postmultiplier $\{k\}$. Combining Eqs. (3) and (4), one has

$$\{p\} = [V][B]^{-1}\{\Delta\} \tag{5}$$

Hence, the desired stiffness matrix $[S]$ is established as

$$[S] = [V][B]^{-1} \tag{6}$$

To develop the second alternative, begin with the virtual work definition of an individual stiffness coefficient[3]

$$S_{ij} = \int_V \lfloor \mathbf{\delta}_i \rfloor \{\mathbf{\varepsilon}_j\} dV \tag{7}$$

where $\lfloor \mathbf{\delta}^i \rfloor$ is the stress vector due to a *unit* displacement at i, $\{\mathbf{\varepsilon}_j\}$ is the strain vector due to a *unit* displacement at j, and dV is a differential volume of the element whose total volume is V. Stress-strain relations can be written in the form

$$\{\mathbf{\varepsilon}\} = [N]\{\mathbf{\delta}\} \tag{8}$$

so that (7) becomes

$$S_{ij} = \int_V \lfloor \mathbf{\delta}_i \rfloor [N]\{\mathbf{\delta}_j\} dV \tag{7a}$$

Combining (1a) with (3) and designating $\{\mathbf{\Delta}_j\}$ as a column vector with 1.0 in the j place and zeros elsewhere gives

$$\{\mathbf{\delta}_j\} = [U][B]^{-1}\{\mathbf{\Delta}_j\} \tag{9}$$

and, through transposition, with the superscript T signifying the transpose of the indicated matrices,

$$\{\mathbf{\delta}_i\} = \lfloor \mathbf{\Delta}_i \rfloor ([B]^{-1})^T [U]^T \tag{9a}$$

Hence, one can write (7a) as (the $[B]$ matrix is unaffected by the integration)

$$S_{ij} = \lfloor \mathbf{\Delta}_i \rfloor ([B]^{-1})^T \left[\int_V [U]^T[N][U] dV\right] [B]^{-1}\{\mathbf{\Delta}_j\} \tag{7b}$$

By direct reasoning, it follows that

$$[S] = ([B]^{-1})^T[G][B]^{-1} \tag{10}$$

where

$$[G] = \left[\int_V [U]^T[N][U] dV\right] \tag{11}$$

Summarizing the formations established here and in Ref. 1, one has the following:

1) From virtual work (virtual forces), indirectly, via the derivation of the flexibility matrix and inversion thereof,

$$[\bar{S}] = [\bar{V}][\bar{G}]^{-1}[\bar{V}]^T \tag{12}$$

$$\left([\bar{G}] = \left[\int_V [\bar{U}]^T[N][\bar{U}] dV\right]\right) \tag{13}$$

[As indicated earlier, in connection with Eqs. (1) and (1a), matrices with bars over their designating symbols exclude only the terms associated with rigid body motion of the element.]

2) From "direct formulation,"

$$[S] = [V][B]^{-1} \tag{6}$$

3) From virtual work (virtual displacements) via direct derivation

$$[S] = ([B]^{-1})^T[G][B]^{-1} \tag{10}$$

In the virtual forces approach, the result is dependent upon the transformation matrix $[\bar{V}]$ of the edge stresses into stress resultants; displacement transformations (idealizations) do not appear. Consequently, displacements corresponding to the chosen stress field may violate compatibility. Conversely, the virtual displacement approach depends upon the displacement transformation matrix $[B]$, excludes the stress-to-corner force transformation matrix $[V]$, and can be used in developments based on assumed displacements whose counterpart stresses violate the conditions of equilibrium. The direct formulation, Eq. (6), involves both the force and displacement transformations and nothing else.

All three approaches will yield identical element relationships for simple conditions, e.g., prismatic members governed by elementary flexure, triangular plates under constant plane stress, etc. This does not necessarily hold for complex geometric conditions and assumed behavior representations such as arbitrary quadrilaterals under direct stress or bending. Furthermore, under the latter conditions, a straightforward derivation of a symmetric $[S]$ matrix through use of Eq. (6) may prove difficult to achieve. $[S]$ matrices derived via Eqs. (10) and (12) must be symmetric, since the indicated matrix products are congruent transformations with $[G]$ and $[\bar{G}]$ symmetric matrices. Developments starting with strain energy theorems (Castigliano's first theorem) also lead to Eqs. (10) and (12). Each row of $[G]$, for example, will be an equation for the derivative of the strain energy (expressed in terms of the coefficients k) with respect to one of the coefficients k.

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Supersonic Interference Lift

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It is shown that the interference lift of a body-wing combination proposed and analyzed previously can be higher than originally anticipated. Whitham's theory is adapted to predict the load distribution induced on the wing by the body.

RESEARCH on favorable supersonic interference of lifting wingbody systems has attracted considerable interest in recent years.[1,2] Since it is possible to improve aerodynamic efficiency at supersonic speeds by using the lift

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